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Small Obstacle
and
Aperture Theory

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SMALL OBSTACLE AND APERTURE THEORY

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ABSTRACT

This report is essentially tutorial in nature. Formulas, based
on the small obstacle and small aperture approximation, are
derived for the equivalent network and scattering parameters
for arbitrary discontinuities in single-mode uniform cylindrical
waveguide. A few examples are presented in order to illus-
trate the simplicity of this method.

CONTENTS

SMALL OBSTACLE AND APERTURE THEORY

I.   INTRODUCTION

The determination of the scattering parameters (matrix) or equivalent networks for discontinuities in uniform cylindrical waveguide usually requires the evaluation of an extremely complex boundary value problem. In general, solutions are not easily obtained and four methods which lead to approximate solutions are classified as

(a)   The variational method

(b)   The integral equation method

(c)   The equivalent static method

(d)   The transform method.

The reader is referred to the Waveguide Handbook,[1] Sec. 3.5, for a thorough discussion of the above methods. When the discontinuity is "small," however, it is possible to deal with this class of problem in a straightforward and simple fashion. A small discontinuity is defined as one which is located "far" from the waveguide walls and whose dimensions are small compared with wavelength. When these conditions are satisfied, the scattering of the electromagnetic fields by such discontinuities illuminated by a known excitation is, to a good approximation, independent of the effect of the waveguide walls and depends only on the geometry of the discontinuity and the nature of the illumination. That quantity, which is solely dependent on discontinuity geometry and excitation field, is the polarizability. Since the discontinuity is small compared with wavelength, the phase of the excitation field is essentially constant over the entire discontinuity. The polarizability may therefore be determined from static field considerations, which is a great simplification.

It is the purpose of small obstacle or aperture theory to combine the discontinuity polarizability with object position in the waveguide structure and the incident mode fields in such a way that the scattering matrix and equivalent circuit parameters may be easily determined. It should be noted that not only does this approach provide a simple solution to otherwise complex problems, but also that these solutions are useful in regions where the approximation would not be expected to hold.

The results which are presented below are for arbitrary, lossless, symmetric, and asymmetric discontinuities in dominant mode uniform cylindrical waveguide.

II. MODAL ANALYSIS AND THE REPRESENTATION OF FIELDS PRODUCED BY CURRENTS AND DISCONTINUITIES IN WAVEGUIDES

The basis for this section can be found in the excellent paper by Marcuvitz and Schwinger.[2] Our purpose is to present those results which will lead directly to the small obstacle or aperture equations. Some typical discontinuities are shown below in Fig. 1(a-d).
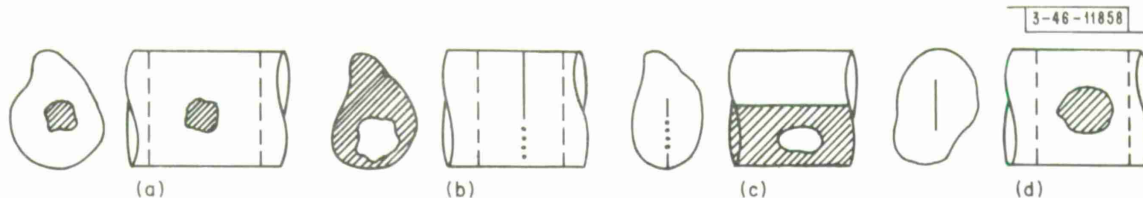

Fig. 1. Typical discontinuities in waveguides: (a) arbitrary obstacle; (b) transverse aperture or obstacle; (c) longitudinal aperture; (d) longitudinal obstacle.

It is a fundamental theorem in electromagnetic field theory that a unique solution for the electric and magnetic fields within any region exists provided the tangential component of either the electric field E or magnetic field H is specified at every point on the surface bounding that region (including discontinuity points). Thus, in the steady state, the electromagnetic fields within a source-free region are solutions of the homogeneous Maxwell field equations

$$\nabla \times E = -j\omega\mu H \tag{1a}$$

$$\nabla \times H = j\omega\epsilon E \tag{1b}$$

where an $e^{j\omega t}$ time dependence is understood, and μ and ϵ are the permeability and dielectric constant of the medium.

For an obstacle discontinuity such as that shown in Fig. 1(a), the region of interest is bounded by the waveguide walls, the obstacle surface, and the far "terminal" surfaces. A far terminal surface (shown by a dashed line) is located at a point sufficiently far from the discontinuity such that only dominant mode fields are considered. The higher-order modes generated, being evanescent, may be considered negligible at that surface. It then remains to solve Eqs. (1) for the electric field in the given region subject to the boundary values of

$$n \times E \quad \text{on the guide walls}$$

$$n \times E \quad \text{on the terminal surfaces}$$

$$H \times n \quad \text{on the obstacle surface} \tag{2}$$

(n = outward unit normal).

For the aperture discontinuities shown in Fig. 1, there are two regions of interest: first, the region bounded by the waveguide walls, the aperture surface (shown by a dotted line), and the far terminal surfaces; second, the corresponding region on the opposite side of the aperture. One now solves Eqs. (1) for the magnetic field in each region subject to the boundary values of

$$n \times E \quad \text{on the guide walls}$$

$$H \times n \quad \text{on the terminal surfaces}$$

$$n \times E \quad \text{on the aperture surface} \quad . \tag{3}$$

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In general, it is extremely difficult to solve the boundary value problems for the regions illustrated in Fig. 1. The difficulties arise because the geometrical shapes of these regions are perturbed by the presence of the obstacle or aperture discontinuities resulting in nonseparable coordinate geometries. Consequently, it becomes most convenient to invoke the concept of "equivalence" and replace the discontinuity structures by equivalent electric and magnetic surface current distributions. This approach leads one to consider the solution of the inhomogeneous Maxwell equations

$$\nabla \times E = -j\omega\mu H - M \tag{4a}$$

$$\nabla \times H = j\omega\epsilon E + J \tag{4b}$$

in a discontinuity-free region, subject to elementary boundary conditions where the fields are produced by the electric and magnetic current densities J and M. This region, a perfectly conducting cylindrical waveguide of arbitrary but uniform cross section, is shown below in Fig. 2(a-b).
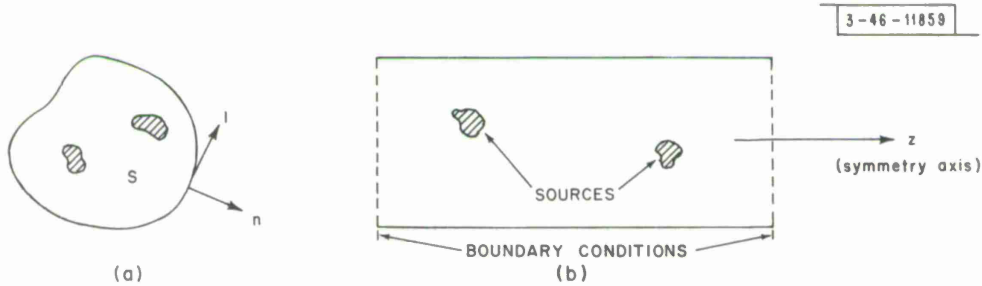

Fig. 2. Discontinuity-free region: (a) transverse view; (b) longitudinal view.

A complete representation of the fields for such regions is required so that solutions of the inhomogeneous field equations [Eqs. (4)] may be obtained. Separation of the total field solutions for E and H into transverse E_t and H_t, and longitudinal E_z and H_z components leads to a great simplification. The inhomogeneous field equations yield for transverse fields

$$-\frac{\partial E_t}{\partial z} = j\omega\mu \, [\mathcal{U} + \nabla_t\nabla_t] \cdot (H_t \times z_o) + (\hat{M}_t \times z_o) \tag{5a}$$

$$-\frac{\partial H_t}{\partial z} = j\omega\epsilon \, [\mathcal{U} + \nabla_t\nabla_t] \cdot (z_o \times E_t) + (z_o \times \hat{J}_t) \tag{5b}$$

and for the longitudinal fields

$$E_z = \frac{z_o}{j\omega\epsilon} \, [\nabla_t \cdot (H_t \times z_o) - J_z] \tag{6a}$$

$$H_z = \frac{z_o}{j\omega\mu} \, [\nabla_t \cdot (z_o \times E_t) - M_z] \tag{6b}$$

where $\mathcal{U}$ is the unit dyadic, $\nabla_t = \nabla - z_o(\partial/\partial z)$ is the transverse gradient operator, and

$$\hat{M}_t = M_t - \frac{1}{j\omega\epsilon} \, \nabla_t \times z_o J_z \tag{7a}$$

$$\hat{J}_t = J_t - \frac{1}{j\omega\mu} \, \nabla_t \times z_o M_z \quad . \tag{7b}$$

The transverse vector field equations [Eqs. (5)] yield the scalar transmission line equations if all transverse vector quantities are represented in terms of a complete set of orthonormal vector mode functions (eigenfunctions). The representation is in the form of a linear superposition of the complete set of eigenfunctions.

$$E_t(r) = \sum_i V_i'(z) \, e_i'(\rho) + \sum_i V_i''(z) \, e_i''(\rho) \tag{8a}$$

$$H_t(r) = \sum_i I_i'(z) \, h_i'(\rho) + \sum_i I_i''(z) \, h_i''(\rho) \tag{8b}$$

and

$$\hat{M}_t(r) = \sum_i v_i'(z) \, h_i'(\rho) + \sum_i v_i''(z) \, h_i''(\rho) \tag{9a}$$

$$\hat{J}_t(r) = \sum_i i_i'(z) \, e_i'(\rho) + \sum_i i_i''(z) \, e_i''(\rho) \tag{9b}$$

where ρ is the cross-sectional vector coordinate, z is the longitudinal (symmetry) coordinate, i is, in general, a double index, and $'$ denotes E-mode and $''$ denotes H-mode. The longitudinal field components are written as

$$j\omega\epsilon E_z(r) = \sum_i I_i'(z) \nabla_t \cdot h_i' \times z_o \tag{10a}$$

$$j\omega\mu H_z(r) = \sum_i V_i''(z) \nabla_t \cdot z_o \times e_i'' \quad . \tag{10b}$$

A few of the more important properties of the eigenfunctions (see Refs. 1 and 2 for more complete descriptions) are:

E-Modes:— The eigenfunction $e_i'(\rho)$ may be derived from a scalar function $\varphi_i(\rho)$ in the following manner

$$e_i'(\rho) = - \frac{\nabla_t \varphi_i(\rho)}{k_{ti}'} \tag{11a}$$

where $\varphi_i(\rho)$ is a solution of

$$(\nabla_t^2 + k_{ti}'^2) \, \varphi_i(\rho) = 0 \tag{11b}$$

subject to the boundary condition $\varphi_i(\rho) = 0$ on l for $k_{ti}' \neq 0$; $[\partial\varphi_i(\rho)/\partial l] = 0$ on l for $k_{ti}' = 0$. In addition,

$$h_i'(\rho) = z_o \times e_i'(\rho) \tag{11c}$$

and

$$e_{zi}'(\rho) = -j \, \frac{Y_i' k_{ti}'^2 \varphi_i(\rho)}{\omega\epsilon} \tag{11d}$$

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where

$$Z_i' = 1/Y_i' = \kappa_i'/\omega\epsilon \quad ; \quad \kappa_i'^2 = k^2 - k_{ti}'^2 \quad .$$

H-Modes:– The eigenfunction $h_i''(\rho)$ may be derived from a scalar function $\psi_i(\rho)$ in the following manner

$$h_i''(\rho) = -\frac{\nabla_t \psi_i(\rho)}{k_{ti}''} \tag{12a}$$

where $\psi_i(\rho)$ is a solution of

$$(\nabla_t^2 + k_{ti}''^2)\,\psi_i(\rho) = 0 \tag{12b}$$

subject to the boundary condition $[\partial\psi_i(\rho)/\partial n] = 0$ on l.   In addition,

$$e_i''(\rho) = h_i''(\rho) \times z_o \tag{12c}$$

and

$$h_{zi}''(\rho) = -j\,\frac{Z_i'' k_{ti}''^2 \psi_i(\rho)}{\omega\mu} \tag{12d}$$

where

$$Z_i'' = 1/Y_i'' = \omega\mu/\kappa_i'' \quad ; \quad \kappa_i''^2 = k^2 - k_{ti}''^2 \quad .$$

The orthogonality properties of the vector and scalar mode functions are

$$\iint_S e_i \cdot e_j^* \, da = \iint_S h_i \cdot h_j^* \, da = \delta_{ij} \tag{13a}$$

$$\iint_S \varphi_i \varphi_j^* \, da = \delta_{ij}/k_{ti}'^2 \quad ; \quad \iint_S \psi_i \psi_j^* \, da = \delta_{ij}/k_{ti}''^2 \tag{13b}$$

where δ_{ij} is the Kronecker delta and the asterisk denotes complex conjugate.

Substitution of Eqs. (8) into Eqs. (5) yields the following generic form for z-dependence of the fields for E- or H-modes:

$$-\frac{dV_i(z)}{dz} = j\kappa_i Z_i I_i(z) + v_i(z) \tag{14a}$$

$$-\frac{dI_i(z)}{dz} = j\kappa_i Y_i V_i(z) + i_i(z) \quad . \tag{14b}$$

Since the vector mode functions form a complete orthonormal set, the source terms $v_i(z)$ and $i_i(z)$ may be found by a simple inversion:

$$v_i(z) = \iint_S \hat{M}_t(r) \cdot h_i^*(\rho) \, da \tag{15a}$$

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$$i_i(z) = \iint_S \hat{J}_t(r) \cdot e_i^*(\rho)\, da \quad . \tag{15b}$$

Substitution of Eqs. (7) into Eqs. (15) obtains, after some vector algebra,

$$v_i(z) = \iint_S M_t(r) \cdot h_i^*(\rho)\, da + \iint_S \left[\frac{z_o \times \nabla_t J_z(r)}{j\omega\epsilon} \right] \cdot h_i^*(\rho)\, da \tag{16a}$$

$$i_i(z) = \iint_S J_t(r) \cdot e_i^*(\rho)\, da + \iint_S \left[\frac{z_o \times \nabla_t M_z(r)}{j\omega\mu} \right] \cdot e_i^*(\rho)\, da \quad . \tag{16b}$$

The expressions $\nabla_t J_z$ and $\nabla_t M_z$ in the integrand imply that J_z and M_z are differentiable functions. Since there are times when these functions are not differentiable (e.g., point sources), the form of the expression is changed by having the ∇_t operate on the mode function which is always well behaved. Utilizing Green's theorem,

$$\iint_S \nabla_t \cdot (fA)\, da = \oint_C f(A \cdot n)\, dl = \iint_S \nabla_t f \cdot A da + \iint_S f\nabla_t \cdot A da \tag{17}$$

where l is the bounding contour, n is the unit normal to l, $J_z = 0$ on l, $h_i \cdot n = 0$ on l, and noting that

$$z_o \times \nabla_t J_z(r) \cdot h_i^*(\rho) = e_i^*(\rho) \cdot \nabla_t J_z(r) \tag{18a}$$

$$z_o \times \nabla_t M_z(r) \cdot e_i^*(\rho) = -h_i^*(\rho) \cdot \nabla_t M_z(r) \tag{18b}$$

the following expressions for $v_i(z)$ and $i_i(z)$ result:

$$v_i(z) = \iint_S M_t(r) \cdot h_i^*(\rho)\, da - \iint_S \frac{J_z(r)\, \nabla_t \cdot e_i^*(\rho)}{j\omega\epsilon}\, da \tag{19a}$$

$$i_i(z) = \iint_S J_t(r) \cdot e_i^*(\rho)\, da - \iint_S \frac{M_z(r)\, \nabla_t \cdot h_i^*(\rho)}{j\omega\mu}\, da \quad . \tag{19b}$$

Since $\nabla_t \cdot e_i(\rho) = j\omega\epsilon Z_i e_{zi}(\rho)$ and $\nabla_t \cdot h_i(\rho) = j\omega\mu Y_i h_{zi}(\rho)$ from Eqs. (11) and (12), Eqs. (19) yield

$$v_i(z) = \iint_S M(r) \cdot h_i^*(\rho)\, da + Z_i^* \iint_S J(r) \cdot e_{zi}^*(\rho)\, da \tag{20a}$$

$$i_i(z) = \iint_S J(r) \cdot e_i^*(\rho)\, da + Y_i^* \iint_S M(r) \cdot h_{zi}^*(\rho)\, da \quad . \tag{20b}$$

The expressions for the modal source functions v_i and i_i are still general.

III. THE SMALL OBSTACLE APPROXIMATION

Application is now made to the situation where discontinuity dimensions are small compared with wavelength. In this situation, the currents induced on the obstacle are approximately in phase and the predominant effect can be related to static charge distributions which vary according to $e^{j\omega t}$ (the time dependence of the excitation). In first-order small obstacle (or aperture)

theory, these induced currents depend only on the discontinuity geometry and the excitation; the effect of the higher-order multipoles and of the surrounding waveguide walls is neglected. One obtains for the induced electric and magnetic dipoles, Q^e and Q^m, respectively,

$$Q^e(r) = \epsilon \mathcal{P} \cdot E_o^{inc}(r) \tag{21a}$$

$$Q^m(r) = \mu \mathbb{M} \cdot H_o^{inc}(r) \tag{21b}$$

where $E_o^{inc}(r)$ and $H_o^{inc}(r)$ are the incident electric and magnetic fields evaluated at the obstacle center; $\mathcal{P}$ and $\mathbb{M}$ are the static electric and magnetic polarizability dyadics, respectively. In addition, if $\delta(r) \ [= \delta(\rho) \, \delta(z)]$, the Dirac delta function, defines the physical extent of the induced dipoles, the equivalent electric and magnetic current dipole elements are given by

$$J(r) = J\delta(r) = j\omega Q^e(r) \, \delta(r) \tag{22a}$$

$$M(r) = M\delta(r) = j\omega Q^m(r) \, \delta(r) \quad . \tag{22b}$$

Substitution of Eqs. (22) into Eqs. (20) yields

$$v_i(z) = v_i \delta(z) = h_{io}^* \cdot M\delta(z) + Z_i^* J \cdot e_{zio}^* \delta(z) \tag{23a}$$

$$i_i(z) = i_i \delta(z) = e_{io}^* \cdot J\delta(z) + Y_i^* M \cdot h_{zio}^* \delta(z) \tag{23b}$$

where the subscript o denotes evaluation of the eigenfunction at the coordinates of the obstacle center. The transmission line equations [Eqs. (14)] may then be written as

$$-\frac{dV_i(z)}{dz} = j\kappa_i Z_i I_i(z) + v_i \delta(z) \tag{24a}$$

$$-\frac{dI_i(z)}{dz} = j\kappa_i Y_i V_i(z) + i_i \delta(z) \tag{24b}$$

and yield, upon integration between $z = 0^-$ and $z = 0^+$,

$$-[V_i(0^-) - V_i(0^+)] = v_i \tag{25a}$$

$$-[I_i(0^-) - I_i(0^+)] = i_i \quad . \tag{25b}$$

The above equations imply the modal equivalent network shown in Fig. 3.
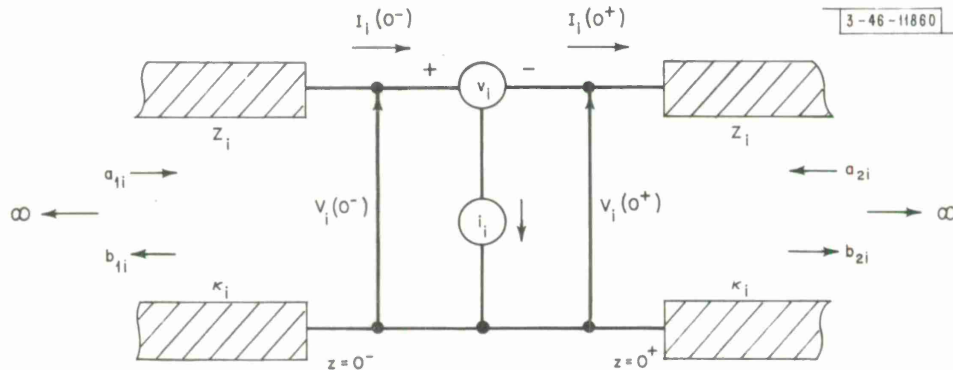

Fig. 3.   Modal equivalent network.

The voltages and currents (V_i and I_i) on the transmission line shown in Fig. 3 are clearly given by

$$V_i(z^{\pm}) = -\frac{1}{2}\left[\pm v_i + Z_i i_i\right] e^{\mp j\kappa_i z} \tag{26a}$$

$$I_i(z^{\pm}) = -\frac{1}{2}\left[\pm i_i + Y_i v_i\right] e^{\mp j\kappa_i z} \quad . \tag{26b}$$

The quantities $V_i(z^{\pm})$ and $I_i(z^{\pm})$ are the scattered voltage and current amplitudes; the mode generators v_i and i_i are related to the incident voltage and current amplitudes. The voltage and current at any point on the transmission line is given in terms of traveling waves,[4]

$$V_i(z) = V_i^{inc} e^{-j\kappa_i z} + V_i^{ref} e^{j\kappa_i z} \tag{27a}$$

$$Z_i I_i(z) = V_i^{inc} e^{-j\kappa_i z} - V_i^{ref} e^{j\kappa_i z} \quad . \tag{27b}$$

The normalized incident and reflected waves as defined by the scattering matrix, $\mathcal{S}(b = \mathcal{S}a)$, are related to the voltage and current in the following manner:

$$a_i = \frac{1}{2}\left[\frac{V_i(z)}{\sqrt{Z_i}} + \sqrt{Z_i}\, I_i(z)\right] \tag{28a}$$

$$b_i = \frac{1}{2}\left[\frac{V_i(z)}{\sqrt{Z_i}} - \sqrt{Z_i}\, I_i(z)\right] \quad . \tag{28b}$$

One therefore obtains the following for a_i and b_i for a two-port structure if the incident energy at ports 1 and 2 are considered separately:

$$a_{1i} = \frac{V_{io}^-}{\sqrt{Z_i}}\, e^{-j\kappa_i z} \tag{29a}$$

$$b_{1i} = \frac{-1}{2\sqrt{Z_i}}\left[-v_i + Z_i i_i\right] e^{j\kappa_i z} \tag{29b}$$

$$a_{2i} = \frac{V_{io}^+}{\sqrt{Z_i}}\, e^{j\kappa_i z} \tag{29c}$$

$$b_{2i} = \frac{-1}{2\sqrt{Z_i}}\left[v_i + Z_i i_i\right] e^{-j\kappa_i z} \quad . \tag{29d}$$

The electromagnetic fields may be concisely described in terms of three-dimensional electric and magnetic vector mode fields associated with the ith mode, having unit voltage and current amplitude, respectively. This leads to a particularly compact form for the scattering parameters. These vector mode fields are

$$\mathcal{E}_i^{(\pm)}(r) = \left[e_i(\rho) \pm e_{zi}(\rho)\right] e^{\mp j\kappa_i z} \tag{30a}$$

$$\mathcal{H}_i^{(\pm)}(r) = [h_i(\rho) \pm h_{zi}(\rho)]\, e^{\mp j\kappa_i z} \tag{30b}$$

where $e_i(\rho)$, $e_{zi}(\rho)$, $h_i(\rho)$, and $h_{zi}(\rho)$ have been defined previously and the unit current is given in terms of the unit voltage as $-Y_i$ for waves traveling in the $-z$ direction and $+Y_i$ for waves traveling in the $+z$ direction. The electric and magnetic dipoles J and M are then given by (the ($\pm$) superscript distinguishes between an equivalent dipole excited by a wave from the right or left, respectively)

$$J^{(\pm)} = j\omega\epsilon\mathcal{P} \cdot \mathcal{E}_{io}^{(\pm)} \tag{31a}$$

$$M^{(\pm)} = \pm j\omega\mu Y_i\mathfrak{M} \cdot \mathcal{H}_{io}^{(\pm)} \quad . \tag{31b}$$

Thus, the scattering parameters describing the behavior of a small obstacle (or longitudinal aperture in the sidewall) of a two-port structure are (note that $Z_i = Z_i^*$ for propagating modes, and $\mathcal{E}_i^{(\pm)*} = \mathcal{E}_i^{(\mp)}$; $\mathcal{H}_i^{(\pm)*} = \mathcal{H}_i^{(\mp)}$ for transversely bounded, lossless waveguide)

$$s_{11} = \frac{j\omega}{2} [\mu Y_i \mathcal{H}_{io}^{(-)*} \cdot \mathfrak{M} \cdot \mathcal{H}_{io}^{(+)} - \epsilon Z_i \mathcal{E}_{io}^{(-)*} \cdot \mathcal{P} \cdot \mathcal{E}_{io}^{(+)}] \tag{32a}$$

$$s_{21} = 1 - \frac{j\omega}{2} [\mu Y_i \mathcal{H}_{io}^{(+)*} \cdot \mathfrak{M} \cdot \mathcal{H}_{io}^{(+)} + \epsilon Z_i \mathcal{E}_{io}^{(+)*} \cdot \mathcal{P} \cdot \mathcal{E}_{io}^{(+)}] \tag{32b}$$

$$s_{12} = 1 - \frac{j\omega}{2} [\mu Y_i \mathcal{H}_{io}^{(-)*} \cdot \mathfrak{M} \cdot \mathcal{H}_{io}^{(-)} + \epsilon Z_i \mathcal{E}_{io}^{(-)*} \cdot \mathcal{P} \cdot \mathcal{E}_{io}^{(-)}] \tag{32c}$$

$$s_{22} = \frac{j\omega}{2} [\mu Y_i \mathcal{H}_{io}^{(+)*} \cdot \mathfrak{M} \cdot \mathcal{H}_{io}^{(-)} - \epsilon Z_i \mathcal{E}_{io}^{(+)*} \cdot \mathcal{P} \cdot \mathcal{E}_{io}^{(-)}] \quad . \tag{32d}$$

The above formulas will yield the scattering parameters for small obstacles, longitudinal, transverse or of arbitrary shape in dominant mode cylindrical waveguide. The scattering parameters for small longitudinal apertures may also be obtained if one replaces the object obstacle polarizabilities $\mathcal{P}$ and $\mathfrak{M}$, by its dual-aperture polarizabilities $\hat{\mathcal{P}}$ and $\hat{\mathfrak{M}}$, which can be shown to have values equal to one-quarter the magnetic and electric obstacle polarizabilities. Thus,

$$\hat{\mathcal{P}} = \frac{1}{4}\mathfrak{M} \tag{33a}$$

$$\hat{\mathfrak{M}} = \frac{1}{4}\mathcal{P} \quad . \tag{33b}$$

On the other hand, the scattering parameters for small transverse apertures can be determined most easily from the equivalent circuit of a small transverse obstacle of the same shape, size, and location via Babinet's principle. This approach will be carried out in Sec. IV. The equivalent circuits for small obstacles will now be deduced from the scattering parameters derived above.

Any two-port (two-terminal-pair) network, such as our small obstacle (or aperture) configurations, may be represented most generally by either a Pi (admittance formulation) or a Tee (impedance formulation) equivalent circuit [see Fig. 4(a-b)].

The relationship between the scattering matrix $\mathcal{S}$ and the impedance and admittance matrices, $\mathcal{Z}$ and $\mathcal{Y}$, for a circuit is expressed below

$$\mathcal{Z} = \mathcal{Y}^{-1} = (1 + \mathcal{S})(1 - \mathcal{S})^{-1} \quad . \tag{34}$$
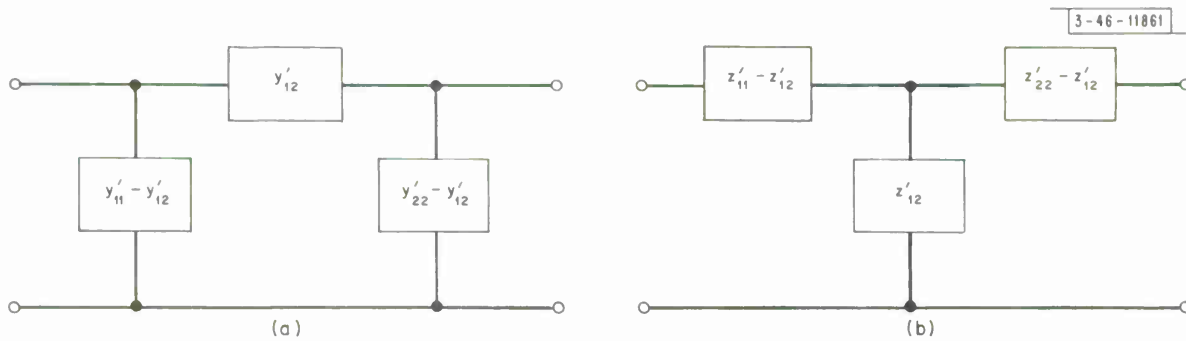
Fig. 4. Equivalent circuits: (a) Pi, and (b) Tee.

Thus, for the Pi and Tee networks shown in Fig. 4(a-b),

$$y'_{11} - y'_{12} = \frac{s_{22} - s_{11} + (1 + s_{12})^2 - s_{11}s_{22}}{\Delta_y} \tag{35a}$$

$$y'_{22} - y'_{12} = \frac{s_{11} - s_{22} + (1 + s_{12})^2 - s_{11}s_{22}}{\Delta_y} \tag{35b}$$

$$y'_{12} = -\frac{2s_{12}}{\Delta_y} \tag{35c}$$

where

$$\Delta_y = 1 + s_{11} + s_{22} + s_{11}s_{22} - s_{12}^2 \tag{35d}$$

$$z'_{11} - z'_{12} = \frac{s_{11} - s_{22} + (1 - s_{12})^2 - s_{11}s_{22}}{\Delta_z} \tag{36a}$$

$$z'_{22} - z'_{12} = \frac{s_{22} - s_{11} + (1 - s_{12})^2 - s_{11}s_{22}}{\Delta_z} \tag{36b}$$

$$z'_{12} = \frac{2s_{12}}{\Delta_z} \tag{36c}$$

where

$$\Delta_z = 1 - s_{11} - s_{22} + s_{11}s_{22} - s_{12}^2 \quad . \tag{36d}$$

In the small obstacle (or aperture) limit, it is easily shown for <u>symmetrical</u> ($y'_{11} = y'_{22}$; $z'_{11} = z'_{22}$) discontinuities that

$$z'_{11} - z'_{12} = \frac{1}{2y'_{12}} \tag{37a}$$

$$y'_{11} - y'_{12} = \frac{1}{2z'_{12}} \quad . \tag{37b}$$

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For small obstacles, it follows directly from Eqs. (32), (35), and (36) that

$$y'_{12} \approx -\frac{1}{s_{11} + (1 - s_{12})}$$ (38a)

$$z'_{12} \approx \frac{1}{(1 - s_{12}) - s_{11}} \quad .$$ (38b)

Upon substituting Eqs. (30), (32a), and (32c) into Eqs. (38), one obtains

$$y'_{12} \approx -\frac{1}{j\omega \, [\mu Y_i h^*_{io} \cdot \mathfrak{M} \cdot h_{io} + \epsilon Z_i e^*_{zio} \cdot \mathcal{P} \cdot e_{zio}]}$$ (39a)

$$z'_{12} \approx \frac{1}{j\omega \, [\mu Y_i h^*_{zio} \cdot \mathfrak{M} \cdot h_{zio} + \epsilon Z_i e^*_{io} \cdot \mathcal{P} \cdot e_{io}]} \quad .$$ (39b)

IV. THE SMALL APERTURE APPROXIMATION

The scattering and equivalent network parameters for small apertures in an infinitely thin, perfectly conducting screen transverse to the symmetry (longitudinal) axis may be obtained directly from the small obstacle results via Babinet's principle, the bisection theorem, and the duality properties of the Maxwell field equations. It should be first realized that a transverse obstacle may be represented by a shunt equivalent circuit. This fact is easily demonstrated by an examination of Eqs. (39). Since the transverse components of the magnetic polarizability dyadic and the normal component of the electric polarizability dyadic are zero for an infinitely thin obstacle, it follows that

$$z'_{11} - z'_{12} = \frac{1}{2y'_{12}} = 0$$ (40a)

$$Y'_{obs} = \frac{1}{Z'_{obs}} = 2(y'_{11} - y'_{12}) = \frac{1}{z'_{12}} = j\omega \, [\mu Y_i h^*_{zio} \cdot \mathfrak{M} \cdot h_{zio} + \epsilon Z_i e^*_{io} \cdot \mathcal{P} \cdot e_{io}] \quad .$$ (40b)

Figure 5(a-b) shows the equivalent circuit of a thin transverse obstacle.
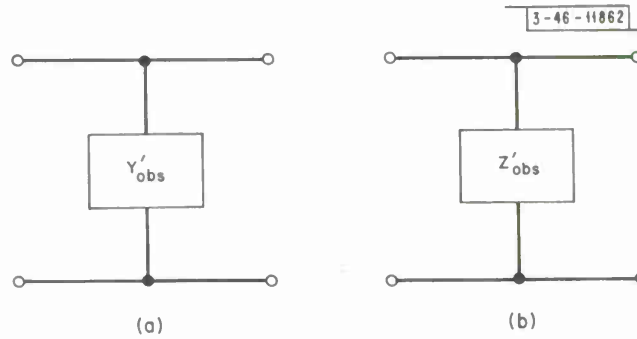

Fig. 5. Equivalent circuit of thin transverse obstacle:
(a) admittance description; (b) impedance description.

It can be shown[5] that the impedance and admittance for the shunt obstacle and aperture are related in the following manner:

$$Z'_{ap} = \frac{1}{Y'_{ap}} = \frac{1}{4} \, Y'_{obs} \quad .$$ (41)

Therefore,

$$Z'_{ap} = j \frac{\omega}{4} [\epsilon Z_i e^*_{zio} \cdot \mathbb{M} \cdot e_{zio} + \mu Y_i h^*_{io} \cdot \mathcal{P} \cdot h_{io}] \tag{42}$$

or, in terms of the aperture polarizabilities $\hat{\mathbb{M}}$ and $\hat{\mathcal{P}}$ [see Eqs. (33)],

$$Z'_{ap} = j\omega [\mu Y_i h^*_{io} \cdot \hat{\mathbb{M}} \cdot h_{io} + \epsilon Z_i e^*_{zio} \cdot \hat{\mathcal{P}} \cdot e_{zio}] \quad . \tag{43}$$

The scattering parameters $s_{11} \left(= |s_{11}| e^{j\varphi_{11}} \right)$ and $s_{12} \left(= |s_{12}| e^{j\varphi_{12}} \right)$ are then defined by

$$s_{11} = - \frac{1}{1 + 2Z'_{ap}} \tag{44a}$$

and, for lossless, reciprocal discontinuities,

$$s_{12}^2 = - (1 - |s_{11}|^2)^2 e^{2j\varphi_{11}} \quad . \tag{44b}$$

All equations derived herein are consistent and exact in the limit of discontinuities of zero extent. However, it has been shown[1,6] that these equations provide useful data even for discontinuities whose dimensions are significant fractions of a wavelength.

V. SELECTED EXAMPLES

The following examples are given to illustrate the ease with which the properties of otherwise difficult boundary value problems may be obtained using small obstacle (aperture) theory.

Example 1

Consider an elliptic aperture in rectangular waveguide with an incident H_{10} mode (see Fig. 6).
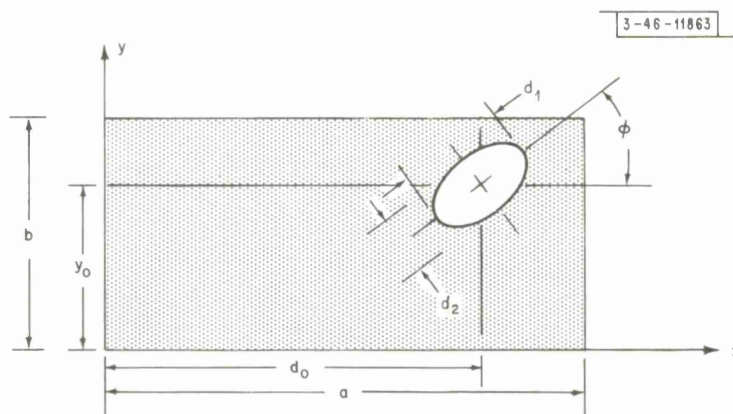

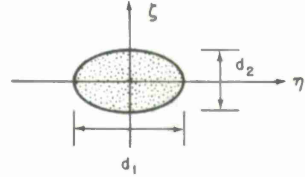Fig. 6. Elliptic aperture in rectangular waveguide with incident H_{10} mode.

From Eq. (43),

$$Z'_{ap} = j\omega\mu Y [\hat{\mathbb{M}}_{xx} |h_{xo}|^2 + \hat{\mathbb{M}}_{yy} |h_{yo}|^2] + j\omega\epsilon Z \hat{\mathcal{P}}_{zz} |e_{zo}|^2$$

where

$$h_x = x_o \sqrt{\frac{2}{ab}} \sin \frac{\pi x}{a} \quad ; \quad e_z = h_y = 0 \quad ; \quad Y = \frac{\kappa}{\omega\mu} \quad ; \quad \kappa = \frac{2\pi}{\lambda_g}$$

and, for an elliptical disk ($d_2 \ll d_1$),

$$\mathbb{M}_{\zeta\zeta} = \frac{\pi}{6} d_1^3 \frac{1}{\ln\left(4\frac{d_1}{d_2} - 1\right)}$$

$$\mathbb{M}_{\eta\eta} = \frac{\pi}{6} d_1 d_2^2 \quad .$$

Therefore,

$$\mathbb{M}_{xx} = \mathbb{M}_{\zeta\zeta} \sin^2\varphi + \mathbb{M}_{\eta\eta} \cos^2\varphi = 4\hat{\mathbb{M}}_{xx} \quad .$$

Finally,

$$Z'_{ap} = j \frac{\pi^2}{6ab\lambda_g} \left[d_1^3 \frac{\sin^2\varphi}{\ln\left(4\frac{d_1}{d_2} - 1\right)} + d_1 d_2^2 \cos^2\varphi \right] \sin^2 \frac{\pi d_o}{a} \quad .$$

Example 2

Consider a narrow centered strip in rectangular waveguide operating in the dominant H_{10} mode only (see Fig. 7).
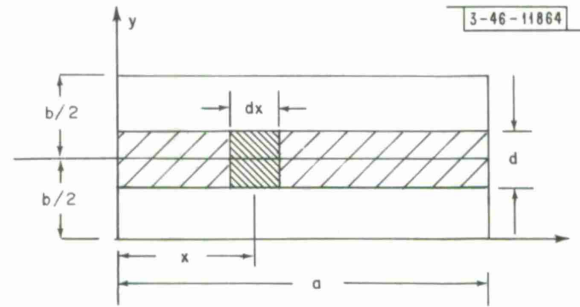

Fig. 7. Narrow centered strip in rectangular waveguide operating in dominant H_{10} mode only.

From Eq. (40b),

$$Y'_{obs} = j\omega\mu Y \mathbb{M}_{zz} |h_{zo}|^2 + j\omega\epsilon Z [\mathcal{P}_{xx} |e_{xo}|^2 + \mathcal{P}_{yy} |e_{yo}|^2]$$

where

$$Z = Y^{-1} = \frac{\omega\mu}{\kappa} \quad ; \quad e_y = \sqrt{\frac{2}{ab}} \sin \frac{\pi x}{a} \quad ; \quad h_z = -j \frac{\pi}{a\kappa} \sqrt{\frac{2}{ab}} \cos \frac{\pi x}{a}$$

$$\mathcal{P}_{yy} = -\mathbb{M}_{zz} = \frac{\pi d^2}{4}/\text{unit length} \quad .$$

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Therefore,

$$dY'_{obs} = j \, \frac{k^2}{\kappa} \left(\frac{\pi d^2}{4}\right) \left(\frac{2}{ab}\right) \sin^2 \frac{\pi x}{a} \, dx - j \, \frac{1}{\kappa} \left(\frac{\pi d^2}{4}\right) \left(\frac{2}{ab}\right) \cos^2 \frac{\pi x}{a} \, dx \quad .$$

Hence, upon integrating from $x = 0$ to $x = a$, one obtains

$$Y'_{obs} = j \, \frac{2b}{\lambda_g} \left(\frac{\pi d}{2b}\right)^2 \quad .$$

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13. ABSTRACT

This report is essentially tutorial in nature. Formulas, based on the small obstacle and small aperture approximation, are derived for the equivalent network and scattering parameters for arbitrary discontinuities in single-mode uniform cylindrical waveguide. A few examples are presented in order to illustrate the simplicity of this method.

14. KEY WORDS

waveguide theory	equivalent networks	dipoles
small obstacle	electromagnetic fields	scattering
aperture		